Zero Knowledge Proofs

SNARKs via Interactive Proofs

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Recall: What is a SNARK ?

- **SNARK**: a <u>succinct</u> proof that a certain statement is true

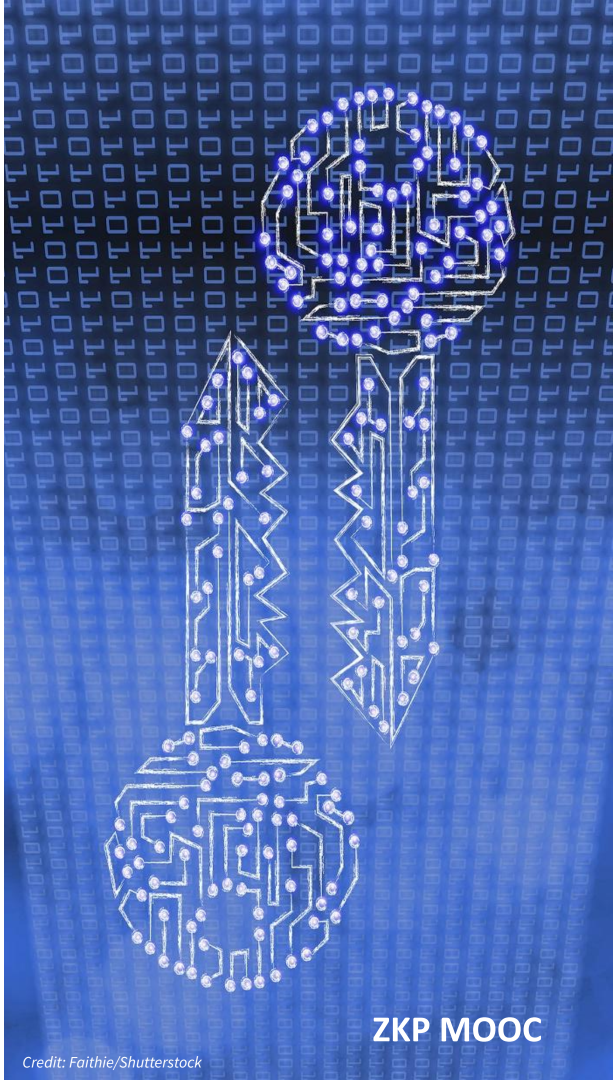
Example statement: "I know an m such that SHA256$(m) = 0$"

- **SNARK**: the proof is **"short"** and **"fast"** to verify

  [if m is 1GB then the trivial proof (the message m) is neither]

zk-SNARK: the proof "reveals nothing" about m (privacy for m)

Interactive Proofs: Motivation and Model



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Interactive Proofs

Cloud Provider

Business/Agency/Scientist

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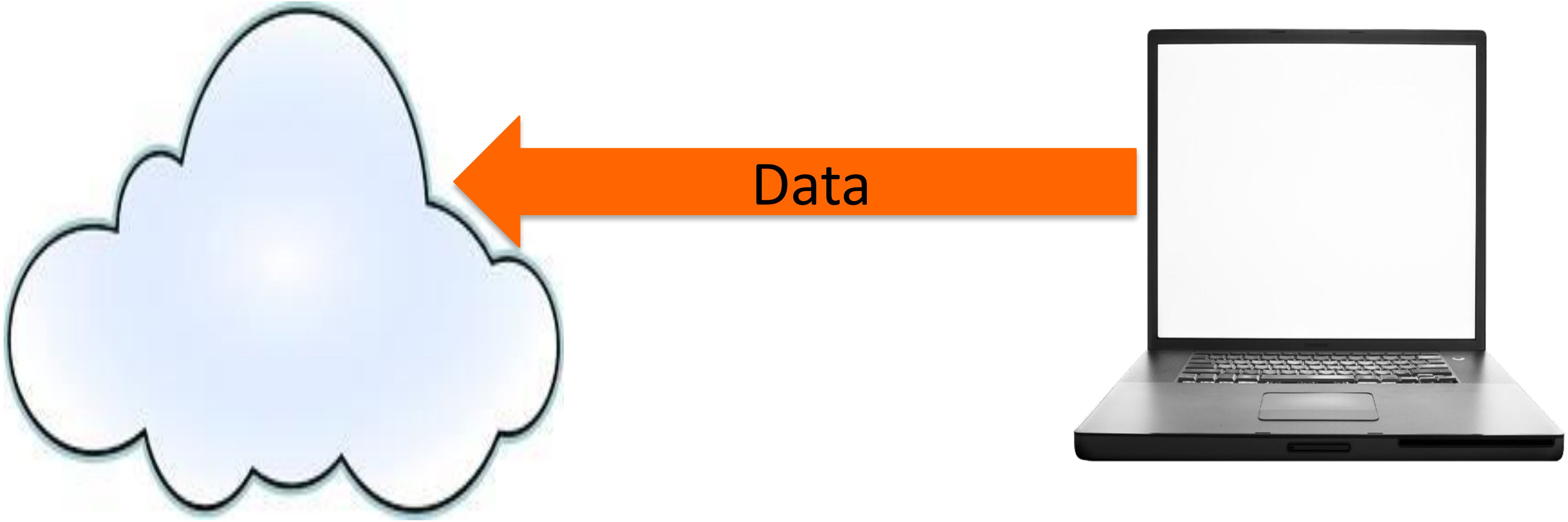

Data

Interactive Proofs

Cloud Provider

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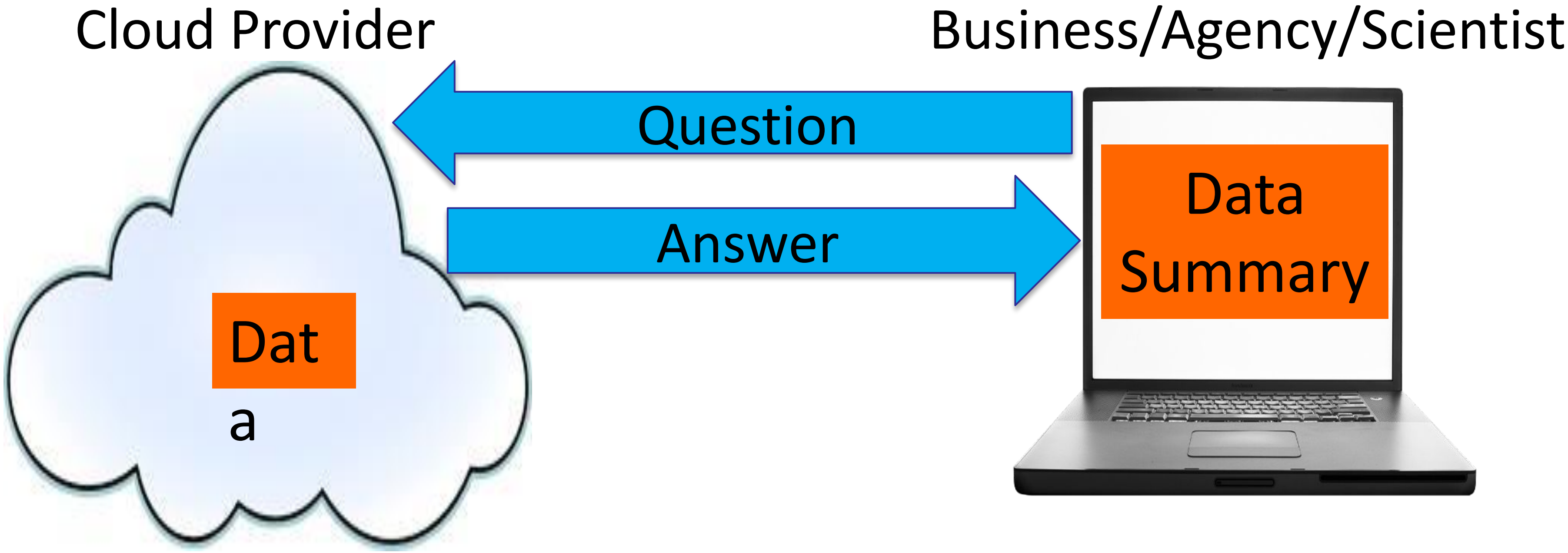
Data
Summary

Interactive Proofs



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Question

Answer

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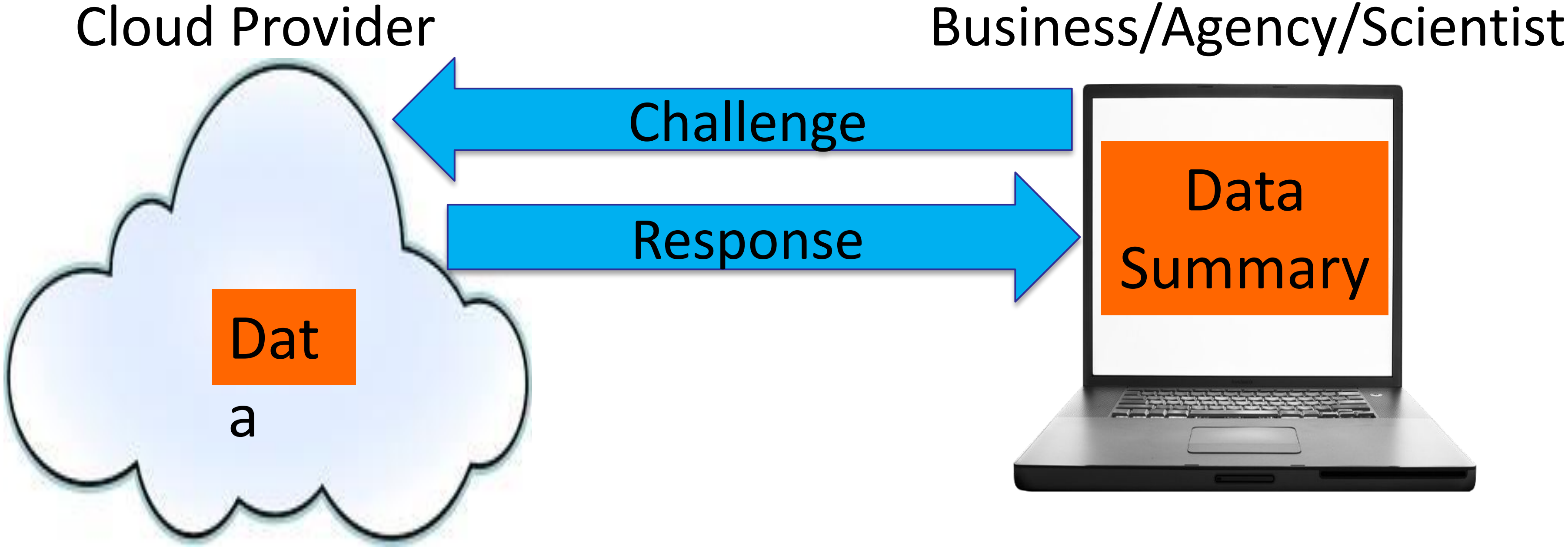
Data Summary

Interactive Proofs



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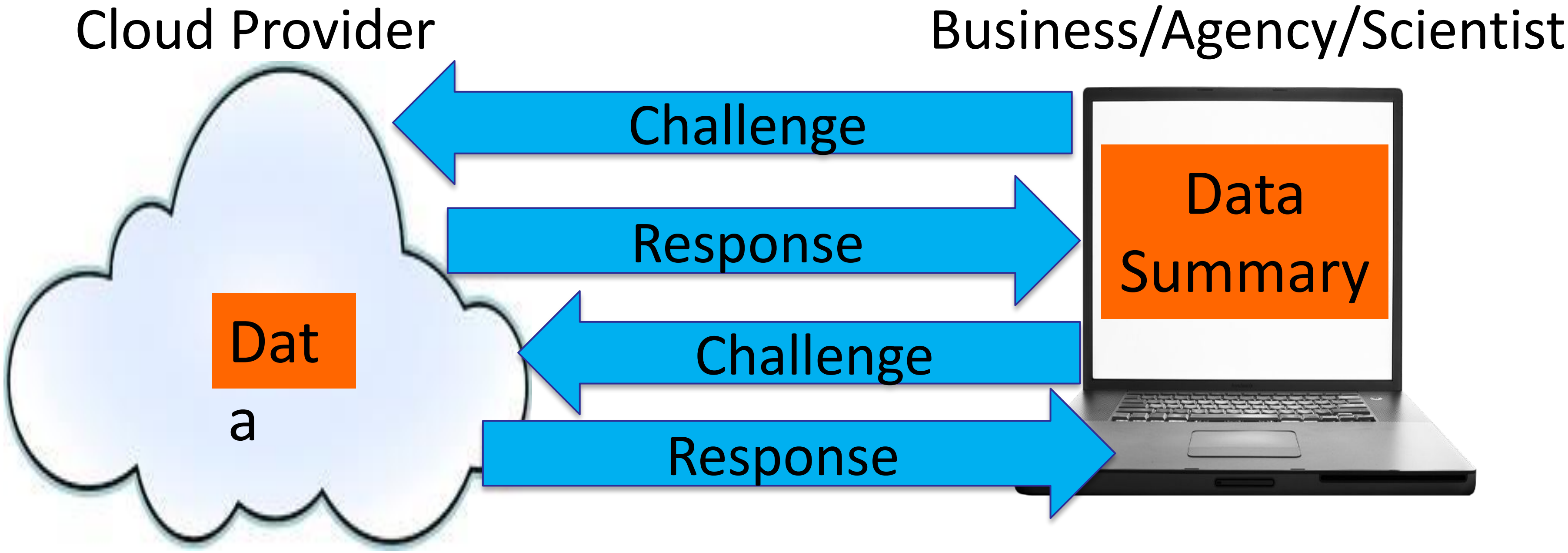

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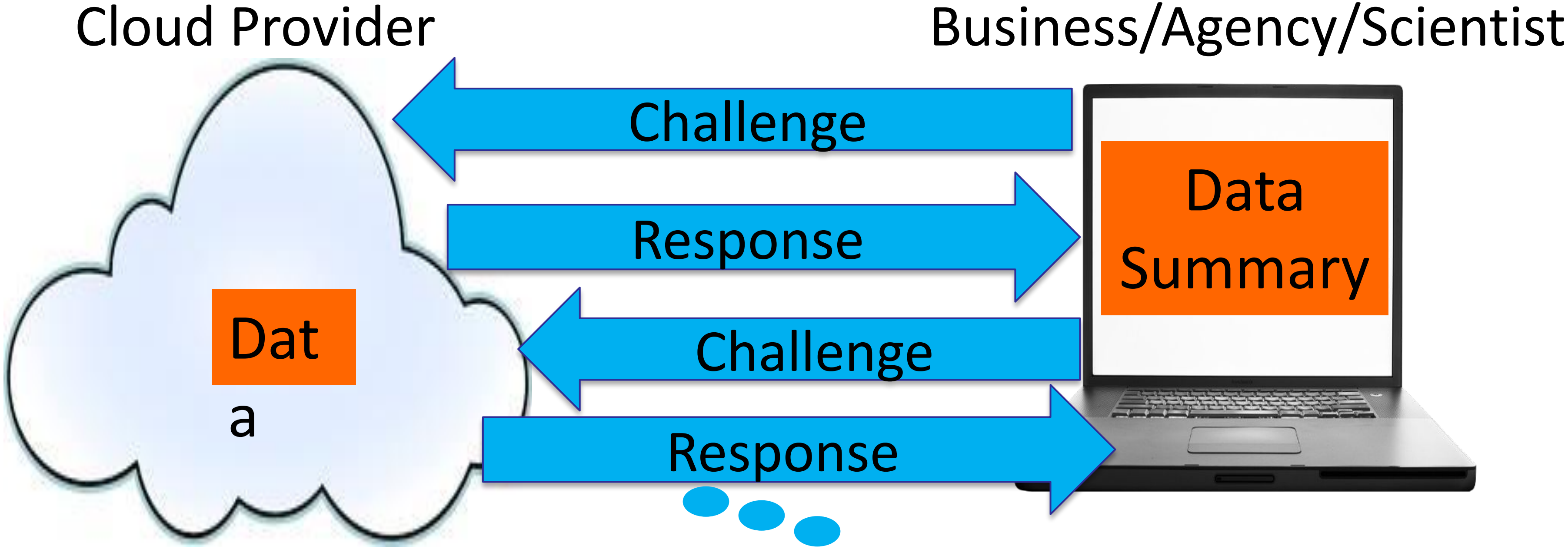

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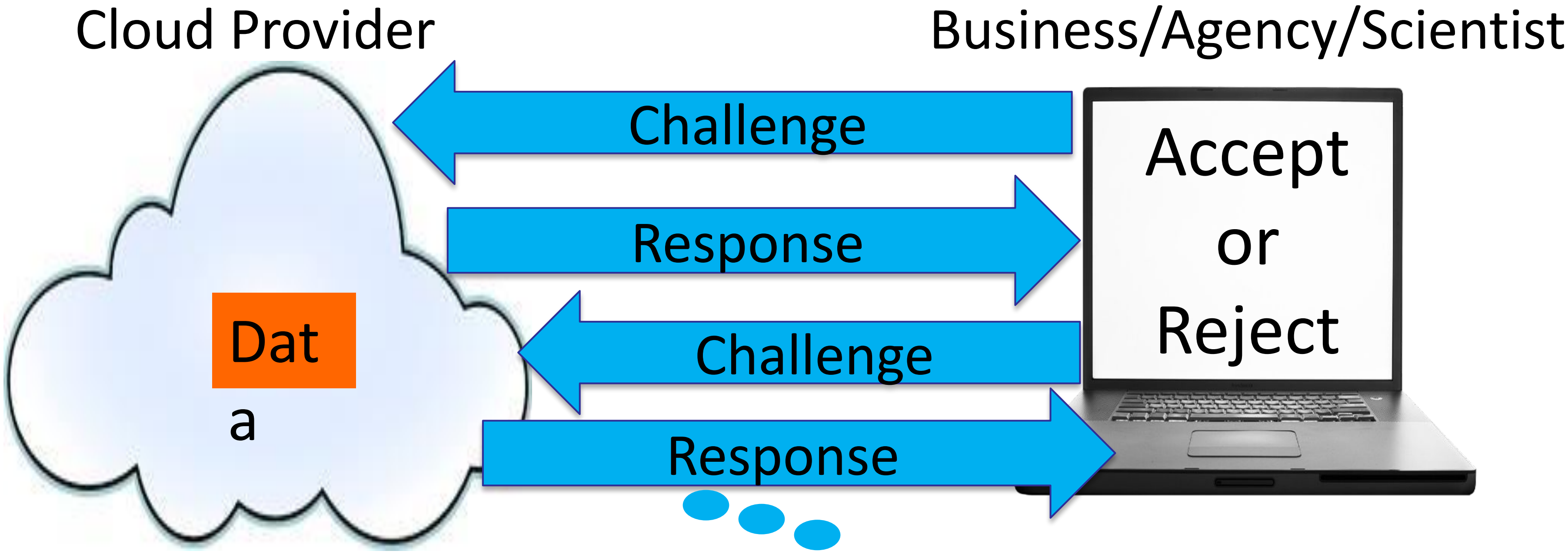

Interactive Proofs

- P solves problem, tells V the answer.
    - Then they have a conversation.
    - P's goal: convince V the answer is correct.
- Requirements:
    - 1. Completeness: an honest P can convince V to accept.
    - 2. (Statistical) Soundness: V will catch a lying P with high probability.

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      If soundness holds only against polynomial-time provers, then the protocol is called an interactive **argument.**

Interactive Proofs and Arguments

- Compare **soundness** to **knowledge soundness** (last lecture) for circuit-satisfiability:

Public arithmetic circuit: $C(\boldsymbol{x}, \boldsymbol{w}) \to \mathbb{F}$

public statement in $\mathbb{F}^n$

secret witness in $\mathbb{F}^m$

Interactive Proofs and Arguments

- Compare **soundness** to **knowledge soundness** (last lecture) for circuit-satisfiability:

- **Sound**: V accepts ⇒ There **exists** w s.t. $C(x, w) = 0$
- **Knowledge sound**: V accepts ⇒ P "knows" w s.t. $C(x, w) = 0$
- Knowledge soundness is stronger.
- But standard soundness is meaningful even in contexts where knowledge soundness isn't.
  - Because there's no natural "witness".
  - E.g., P claims the output of V's program on x is 42.

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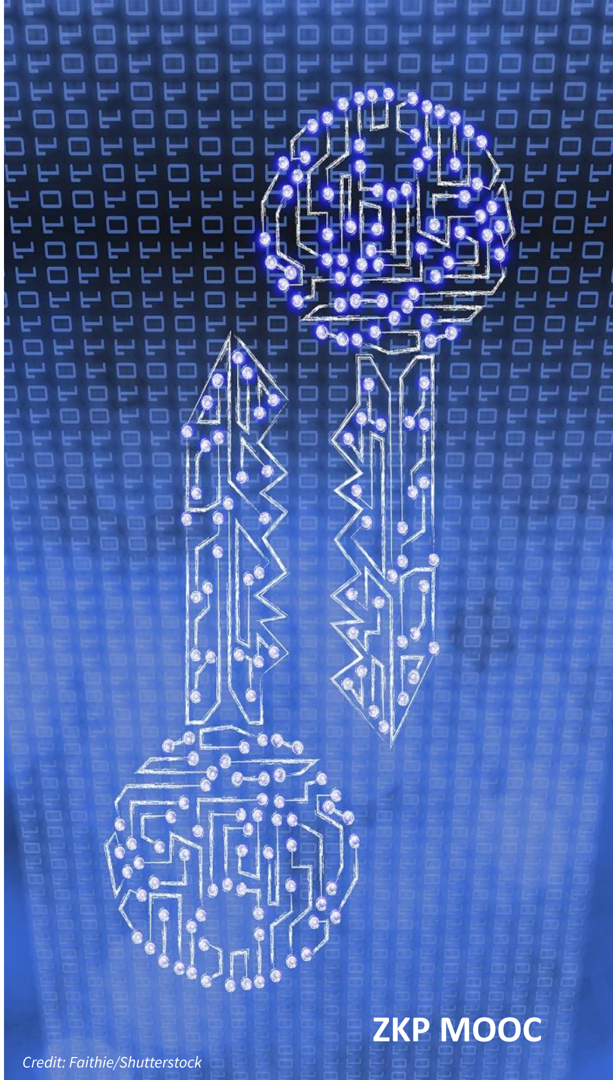
- Knowledge soundness is stronger.
- Likewise, knowledge soundness is meaningful in contexts where standard soundness isn't.
  - e.g., P claims to know the secret key that controls a certain bitcoin wallet.

Public Verifiability

- Interactive proofs and arguments only convince the party that is choosing/sending the random challenges.
- This is bad if there are many verifiers (as in most blockchain applications).
  - P would have to convince each verifier separately.
- For public coin protocols, we have a solution: Fiat-Shamir.
  - Makes the protocol non-interactive + publicly verifiable.

SNARKs from interactive proofs: outline



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Recall: The trivial SNARK is not a SNARK

(a) Prover sends $\boldsymbol{w}$ to verifier,

(b) Verifier checks if $C(\boldsymbol{x}, \boldsymbol{w}) = 0$ and accepts if so.

Problems with this:

(1) $\boldsymbol{w}$ might be long: we want a "short" proof

(2) computing $C(\boldsymbol{x}, \boldsymbol{w})$ may be hard: we want a "fast" verifier

(3) $\boldsymbol{w}$ might be secret: prover might not want to reveal $\boldsymbol{w}$ to verifier

SNARKS from Interactive Proofs (IPs)

- Slightly less trivial: P sends w to V, and uses an IP to prove that w satisfies the claimed property.
  - Fast V, but proof is still too long.
- Actual SNARK: P commits cryptographically to w.
  - Uses an IP to prove that w satisfies the claimed property.
  - Reveals just enough information about the committed witness w to allow V to run its checks in the IP.
  - Render the protocol non-interactive via Fiat-Shamir.

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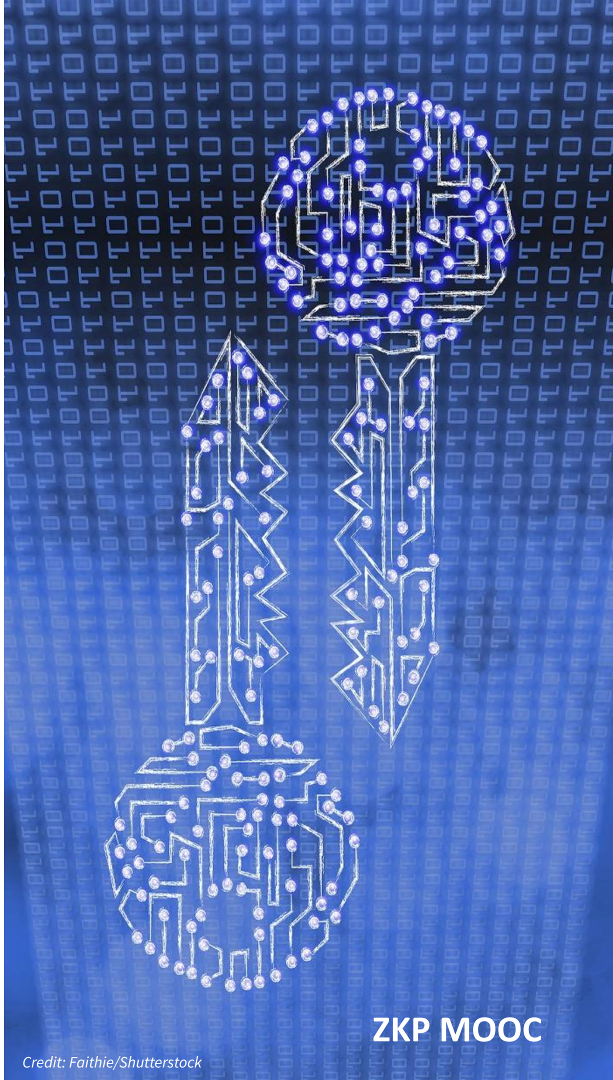
Review of functional commitments



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Recall: three important functional commitments

Polynomial commitments: commit to a univariate $f(X)$ in $\mathbb{F}_p^{(\leq d)}[X]$

Multilinear commitments: commit to multilinear f in $\mathbb{F}_p^{(\leq 1)}[X_1, \ldots, X_k]$

e.g., $f(x_1, \ldots, x_k) = x_1 x_3 + x_1 x_4 x_5 + x_7$

Vector commitments (e.g., Merkle trees):

- Commit to $\vec{u} = (u_1, \ldots, u_d) \in \mathbb{F}_p^d$. Open cells: $f_{\vec{u}}(i) = u_i$

Inner product commitments (inner product arguments – IPA):

Commit to $\vec{u} \in \mathbb{F}_p^d$. Open an inner product: $f_{\vec{u}}(\vec{v}) = (\vec{u}, \vec{v})$

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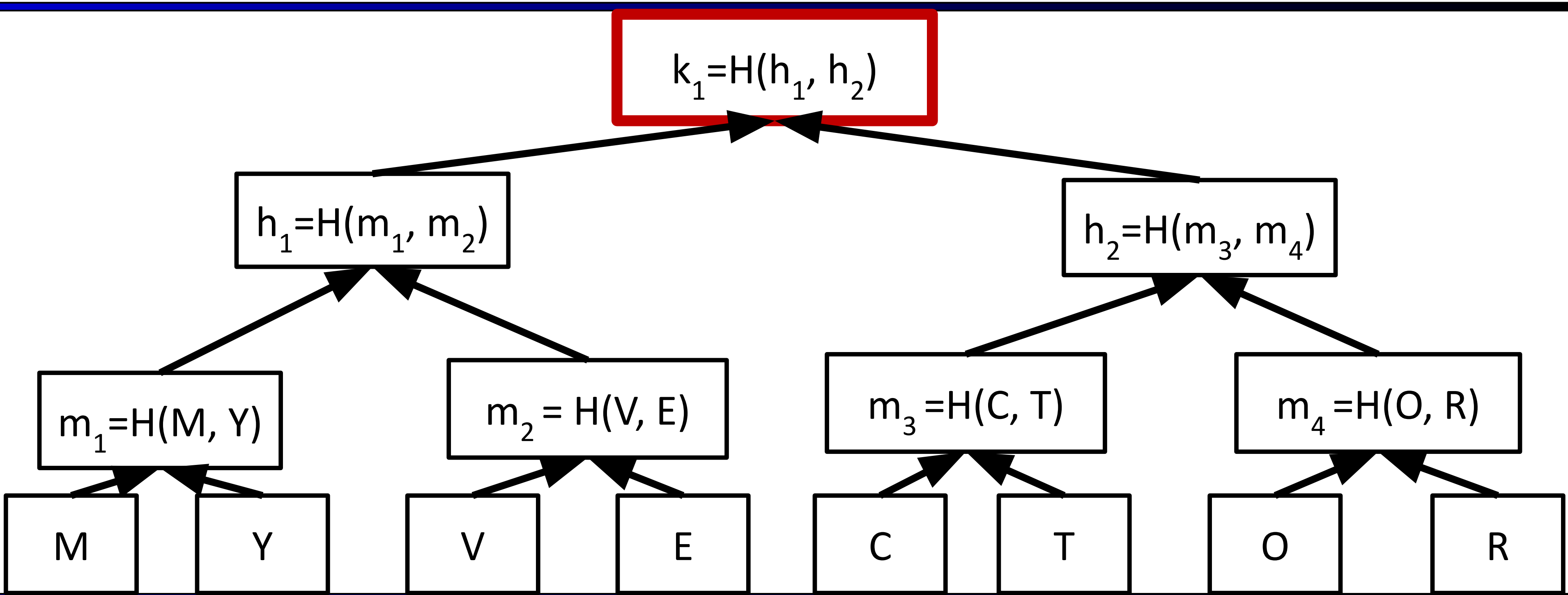
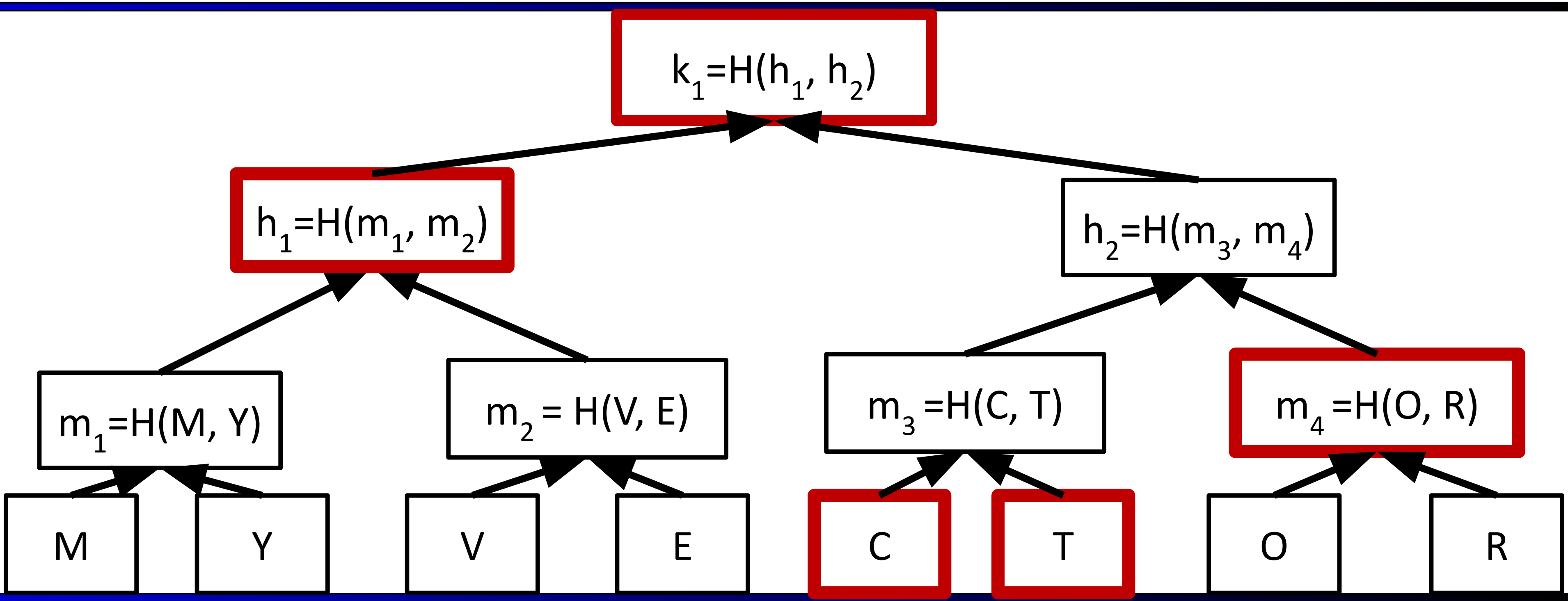
Merkle Trees: The Commitment



Merkle Trees: Opening Leaf T



ZKP MOOC

Merkle Trees

- Commitment to vector is root hash.
- To open an entry of the committed vector (leaf of the tree):
  - Send sibling hashes of all nodes on root-to-leaf path.
  - V checks these are consistent with the root hash.
  - “Opening proof” size is O(log n) hash values.

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- To open an entry of the committed vector (leaf of the tree):
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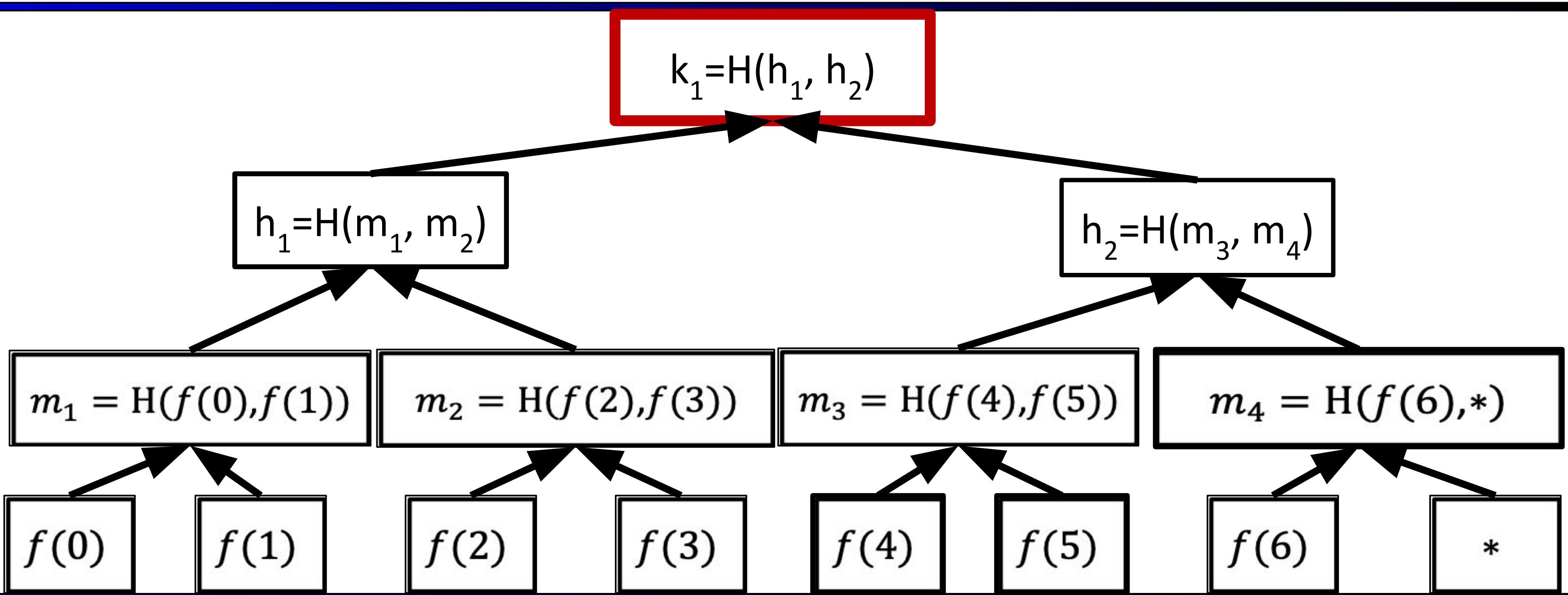
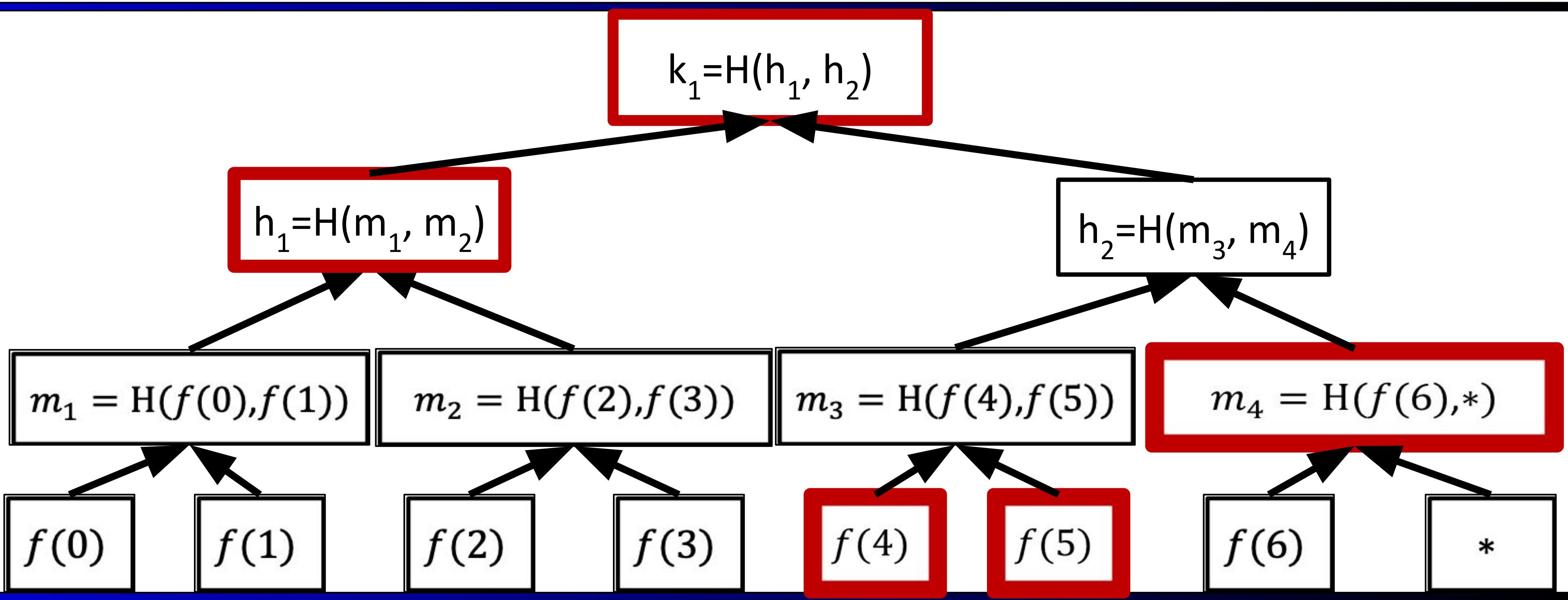
  - V checks these are consistent with the root hash.
  - “Opening proof” size is O(log n) hash values.
- Binding: once the root hash is sent, the committer is bound to a fixed vector.
  - Opening any leaf to two different values requires finding a hash collision (assumed to be intractable).

A First Polynomial commitment: commit to a univariate $f(X)$ in $\mathbb{F}_7^{(\leq d)}[X]$



Reveal $f(4)$



k_1=H(h_1, h_2)
h_1=H(m_1, m_2)
h_2=H(m_3, m_4)
$m_1 = \mathrm{H}(f(0),f(1))$
$m_2 = \mathrm{H}(f(2),f(3))$
$m_3 = \mathrm{H}(f(4),f(5))$
$m_4 = \mathrm{H}(f(6),*)$
$f(0)$
$f(1)$
$f(2)$
$f(3)$
$f(4)$
$f(5)$
$f(6)$
$*$

Summary: commit to a univariate $f(X)$ in $\mathbb{F}^{(\leq d)}[X]$

- P Merkle-commits to all evaluations of the polynomial f.
- When V requests $f(r)$, P reveals the associated leaf along with opening information.

Two problems:

- The number of leaves is $|\mathbb{F}|$, which means the time to compute the commitment is at least $|\mathbb{F}|$.
  - Big problem when working over large fields (say, $|\mathbb{F}| \approx 2^{64}$ or $|\mathbb{F}| \approx 2^{128}$).
  - Want time proportional to the degree bound d.
- V does not know if f has degree at most d!
- We'll explain how to address both issues later in the course.

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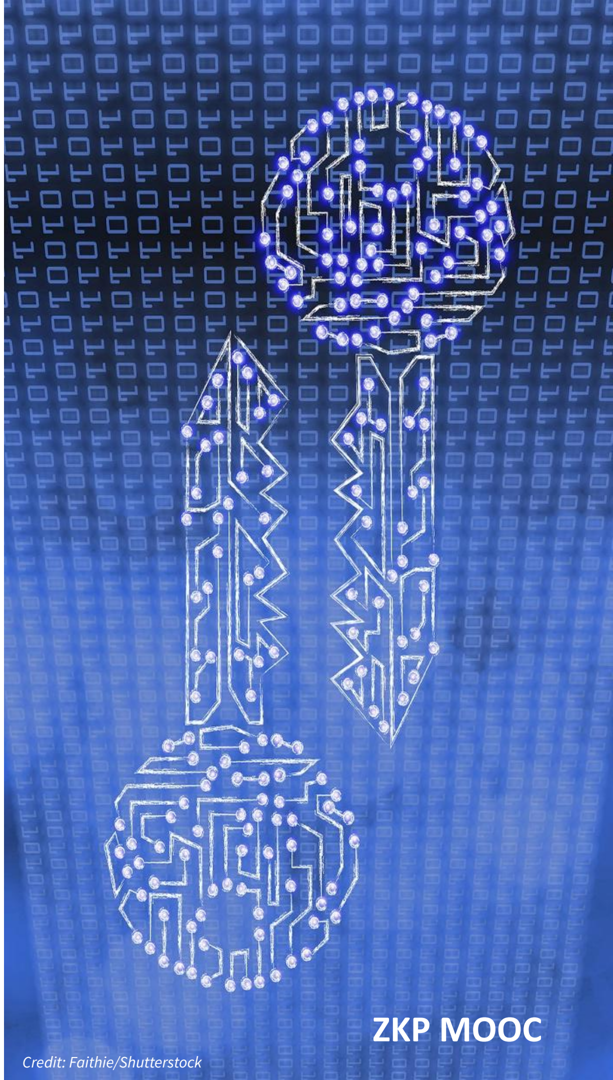
Interactive proof design: Technical preliminaries



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Recap: SZDL Lemma

- Recall **FACT:** Let $p \neq q$ be univariate polynomials of degree at most d. Then $\Pr_{r \in \mathbb{F}}[p(r) = q(r)] \leq \frac{d}{|\mathbb{F}|}$.
- The **Schwartz-Zippel-Demillo-Lipton lemma** is a multivariate generalization:
  - Let $p \neq q$ be ℓ-variate polynomials of total degree at most d. Then $\Pr_{r \in \mathbb{F}^\ell}[p(r) = q(r)] \leq \frac{d}{|\mathbb{F}|}$.
  - "Total degree" refers to the maximum sum of degrees of all variables in any term. E.g., $x_1^2 x_2 + x_1 x_2$ has total degree 3.

Low-Degree and Multilinear Extensions

- Definition [**Extensions**]. Given a function $f\colon \{0,1\}^{\ell} \to \mathbb{F}$, a ℓ-variate polynomial g over $\mathbb{F}$ is said to **extend** f if $f(x) = g(x)$ for all $x \in \{0,1\}^{\ell}$.

- Definition [**Multilinear Extensions].** Any function $f\colon \{0,1\}^{\ell} \to \mathbb{F}$ has a **unique** multilinear extension (MLE), denoted $\tilde{f}$.
  - Multilinear means the polynomial has degree at most 1 in each variable.
  - $(1 - x_1)(1 - x_2)$ is multilinear, $x_1^2 x_2$ is not.

$f : \{0,1\}^2 \rightarrow \mathbb{F}$

1	2
8	10

$\tilde{f}: \mathbb{F}^2 \to \mathbb{F}$

1	2	3	4	5	6
8	10	12	14	16	18
15	18	21	24	27	30
22	26	30	34	38	42
29	34	39	44	49	56

●●●

$$\tilde{f}(x_1, x_2) = (1 - x_1)(1 - x_2) + 2(1 - x_1)x_2 + 8x_1(1 - x_2) + 10x_1x_2$$

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Can check:

$\tilde{f}(0,0) = 1$

$\tilde{f}(0,1) = 2$

$\tilde{f}(1,0) = 8$

$\tilde{f}(1,1) = 10$

Another (non-multilinear) extension of f: $g(x_1, x_2) = -x_1^2 + x_1x_2 + 8\,x_1 + \; x_2 + 1$

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Evaluating multilinear extensions quickly

- Fact: Given as input all 2^ℓ evaluations of a function $f: \{0,1\}^\ell \to \mathbb{F}$, for any point $r \in \mathbb{F}^\ell$ there is an $O(2^\ell)$-time algorithm for evaluating $\tilde{f}(r)$.

Sketch: Use Lagrange interpolation.

Define $\tilde{\delta}_w(r) = \prod_{i=1}^{\ell}(r_i w_i + (1 - r_i)(1 - w_i))$. This is called the mulilinear Lagrange basis polynomial corresponding to w.

Fact: $\tilde{f}(r) = \sum_{w \in \{0,1\}^\ell} f(w) \cdot \tilde{\delta}_w(r)$.

For each $w \in \{0,1\}^\ell$, $\tilde{\delta}_w(r)$ can be computed with $O(\ell)$ field operations.

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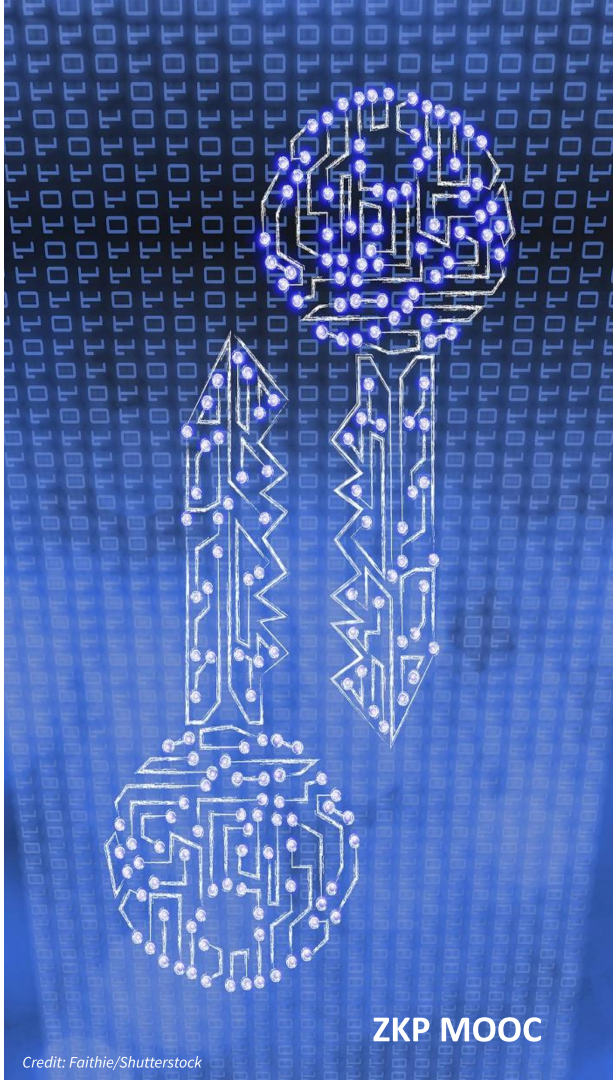
The sum-check protocol



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Sum-Check Protocol [LFKN90]

- Input: V given oracle access to a ℓ-variate polynomial g over field $\mathbb{F}$.
- Goal: compute the quantity:

$$\sum_{b_1 \in \{0,1\}} \sum_{b_2 \in \{0,1\}} \dots \sum_{b_\ell \in \{0,1\}} g(b_1, \dots, b_\ell).$$

Sum-Check Protocol [LFKN90]

- **Start**: P sends claimed answer C_1. The protocol must check that:

$$C_1 = \sum_{b_1\in\{0,1\}} \sum_{b_2\in\{0,1\}} \dots \sum_{b_\ell\in\{0,1\}} g(b_1, \dots, b_\ell).$$

- Round 1: P sends univariate polynomial $s_1(X_1)$ claimed to equal:

$$H_1(X_1) := \sum_{b_2\in\{0,1\}} \dots \sum_{b_\ell\in\{0,1\}} g(X_1, b_2, \dots, b_\ell)$$

V checks that $C_1 = s_1(0) + s_1(1)$.

If this check passes, it is safe for V to believe that C_1 is the correct answer, so long as V believes that $s_1 = H_1$.

How to check this? Just check that s_1 and H_1 agree at a random point r_1.

V can compute $s_1(r_1)$ directly from P's first message, but not $H_1(r_1)$.

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- V checks that $C_1 = s_1(0) + s_1(1)$.
- If this check passes, it is safe for V to believe that C_1 is the correct answer, so long as V believes that $s_1 = H_1$.
- How to check this? Just check that s_1 and H_1 agree at a random point r_1.

V can compute $s_1(r_1)$ directly from P's first message, but not $H_1(r_1)$.

Sum-Check Protocol [LFKN90]

- **Start**: P sends claimed answer C_1. The protocol must check that:

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A i.e., that $s_1(r_1) = \sum_{b_2\in\{0,1\}} \cdots \sum_{b_\ell\in\{0,1\}} g(r_1, b_2, \dots, b_\ell)$.

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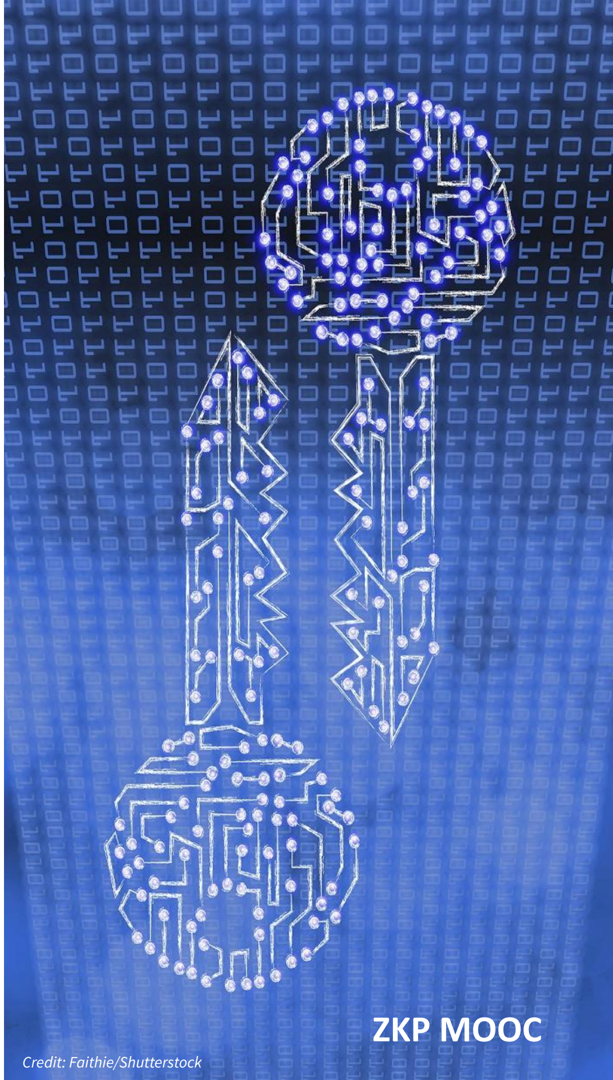
Sum-Check Protocol [LFKN90]

- **Round ℓ (Final round):** P sends univariate polynomial $s_\ell(X_\ell)$ claimed to equal
$$H_\ell := g(r_1, \dots, r_{\ell-1}, X_\ell).$$
- V checks that $s_{\ell-1}(r_{\ell-1}) = s_\ell(0) + s_\ell(1)$.
- V picks r_ℓ at random, and needs to check that $s_\ell(r_\ell) = g(r_1, \dots, r_\ell)$.
  - No need for more rounds. V can perform this check with one oracle query.

Analysis of the sum-check protocol



ZKP MOOC

Credit: Faithie/Shutterstock

Completeness

- Completeness holds by design: If P sends the prescribed messages, then all of V's checks will pass.

Soundness

- If P does not send the prescribed messages, then V rejects with probability at least 1- $\frac{\ell \cdot d}{|\mathbb{F}|}$, where d is the maximum degree of g in any variable.
- E.g. $|\mathbb{F}| \approx 2^{128}, d = 3, \ell \ = 60.$
  - Then soundness error is at most $3 \cdot 60/\ 2^{128}$= 2^{-120}.

Soundness

- If P does not send the prescribed messages, then V rejects with probability at least 1- $\frac{\ell \cdot d}{|\mathbb{F}|}$, where d is the maximum degree of g in any variable.
- Proof is by induction on the number of variables ℓ.
  - Base case: $\ell = 1$. In this case, P sends a single message $s_1(X_1)$ claimed to equal $g(X_1)$. V picks r_1 at random, checks that $s_1(r_1) = g(r_1)$.
  - If $s_1 \neq g$, then $\Pr_{r_1 \in \mathbb{F}}[s_1(r_1) = g(r_1)] \leq \frac{d}{|\mathbb{F}|}$.

Soundness

- Inductive case: $\ell > 1$.
  - Recall: P's first message $s_1(X_1)$ is claimed to equal

    $$H_1(X_1) := \sum_{b_2 \in \{0,1\}} \dots \sum_{b_\ell \in \{0,1\}} g(X_1, b_2, \dots, b_\ell).$$
  - Then V picks a random r_1 and sends r_1 to P. They (recursively) invoke sum-check to confirm that $s_1(r_1) = H_1(r_1)$.
- If $s_1 \neq H_1$, then $\Pr_{r_1 \in \mathbb{F}}[s_1(r_1) = H(r_1)] \le \frac{d}{|\mathbb{F}|}$.
- If $s_1(r_1) \neq H(r_1)$, P is left to prove a false claim in the recursive call.
  - The recursive call applies sum-check to $g(r_1, X_2, \dots, X_\ell)$, which is ℓ-1 variate.
  - By induction, P fails to convince V in the recursive call with probability at least $1 -$

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  - The recursive call applies sum-check to $g(r_1, X_2, \dots, X_\ell)$, which is ℓ-1 variate.
  - By induction, P convinces V in the recursive call with probability at most $\frac{d(\ell-1)}{|\mathbb{F}|}$.

Soundness analysis: wrap-up

- **Summary:** if $s_1 \neq H_1$, the probability V accepts is at most:

$$\Pr_{r_1 \in \mathbb{F}}[s_1(r_1) = H(r_1)] + \Pr_{r_2,\ldots,r_\ell \in \mathbb{F}}[\text{V accepts} | s_1(r_1) \neq H(r_1)]$$

$$\leq \frac{d}{|\mathbb{F}|} + \frac{d(\ell-1)}{|\mathbb{F}|} \leq \frac{d\ell}{|\mathbb{F}|}.$$

Costs of the sum-check protocol

- Total communication is $O(d\ell)$ field elements.
  - P sends ℓ messages, each a univariate polynomial of degree at most d. V sends $\ell - 1$ messages, each consisting of one field element.

V's runtime is:

$$O(d\ell + [\textit{time required to evaluate g at one point}]).$$

P's runtime is at most:

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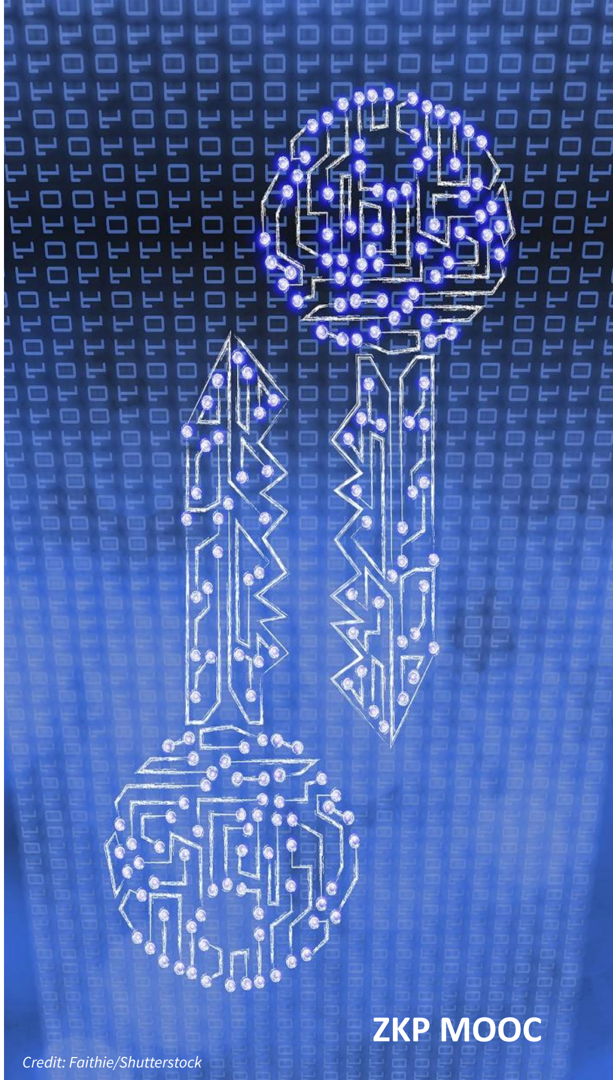
A first application of the sum-check protocol: An IP for counting triangles with linear-time verifier



Credit: Faithie/Shutterstock

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Counting Triangles

- Input: $A \in \{0,1\}^{n\times n}$, representing the adjacency matrix of a graph.
- Desired Output: $\sum_{(i,j,k)\in[n]^3} A_{ij}A_{jk}A_{ik}$.
- Fastest known algorithm runs in matrix-multiplication time, currently about $n^{2.37}$.

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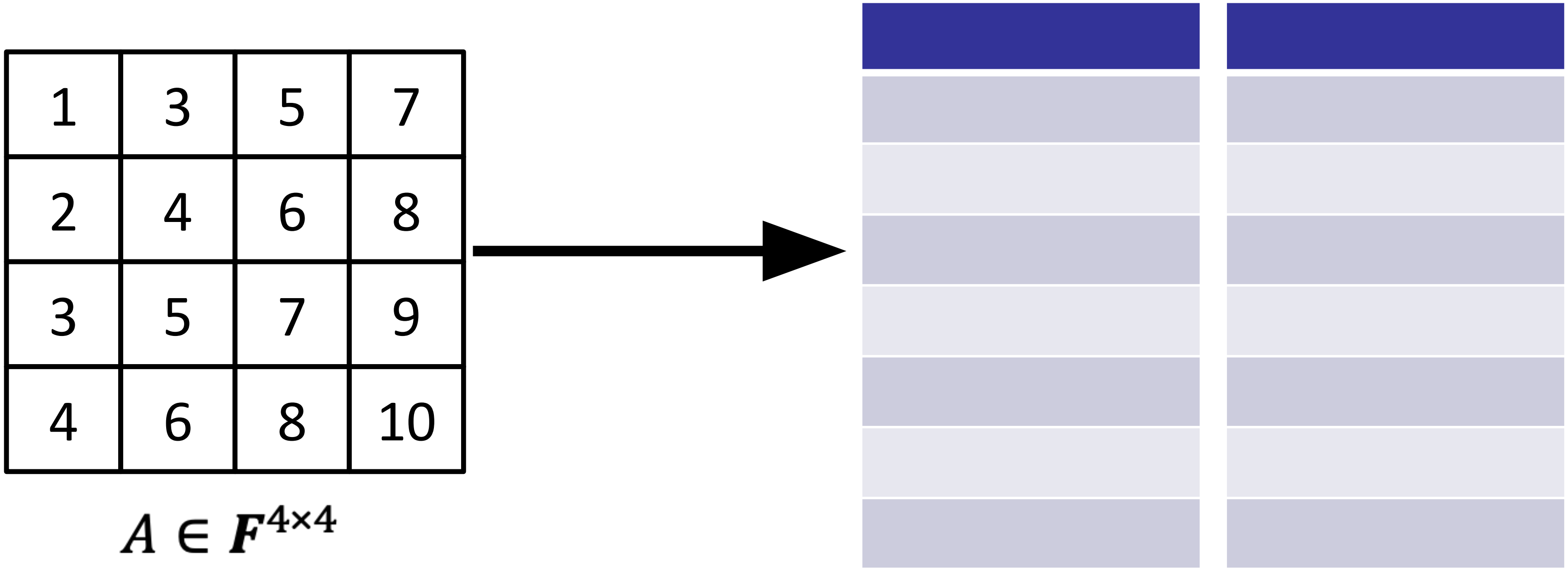
  - View A as a function mapping $\{0,1\}^{\log n} \times \{0,1\}^{\log n}$ to $\mathbb{F}$.



1	3	5	7
2	4	6	8
3	5	7	9
4	6	8	10

$A \in \boldsymbol{F}^{4\times4}$

Counting Triangles

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- Desired Output: $\sum_{(i,j,k)\in[n]^3} A_{ij} A_{jk} A_{ik}$.
- The Protocol:
  - View A as a function mapping $\{0,1\}^{\log n} \times \{0,1\}^{\log n}$ to $\mathbb{F}$.
  - Recall that $\tilde{A}$ denotes the multilinear extension of A.
  - Define the polynomial $g(X,Y,Z) = \tilde{A}(X,Y)\,\tilde{A}(Y,\mathrm{Z})\,\tilde{A}(X,\mathrm{Z})$
  - Apply the sum-check protocol to g to compute:

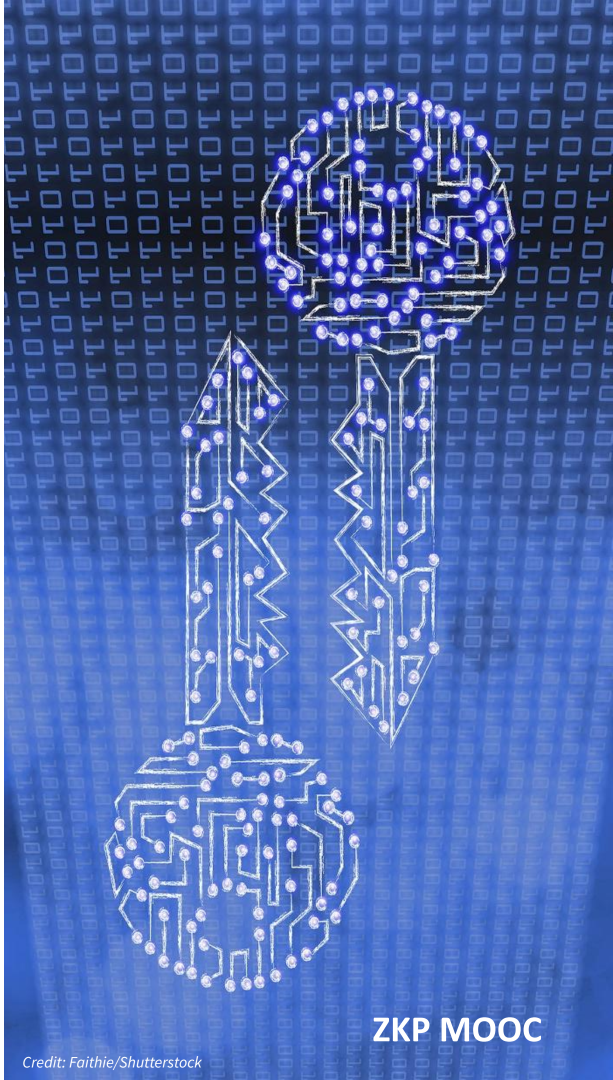
$$\sum_{(a,b,c)\,\in\{0,1\}^{3\log n}} g(a,b,c)$$

Counting Triangles

- Costs:
  - Total communication is $O(\log n)$, V runtime is $O(n^2)$, P runtime is $O(n^3)$.
  - V's runtime dominated by evaluating:

$$g(r_1, r_2, r_3) = \tilde{A}(r_1, r_2)\ \tilde{A}(r_2, r_3)\ \tilde{A}(r_1, r_3).$$

A SNARK for circuit-satisfiability



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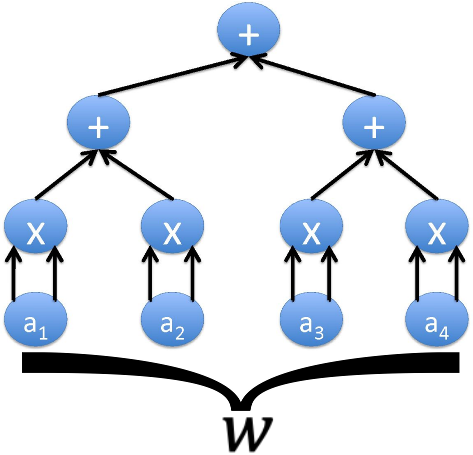
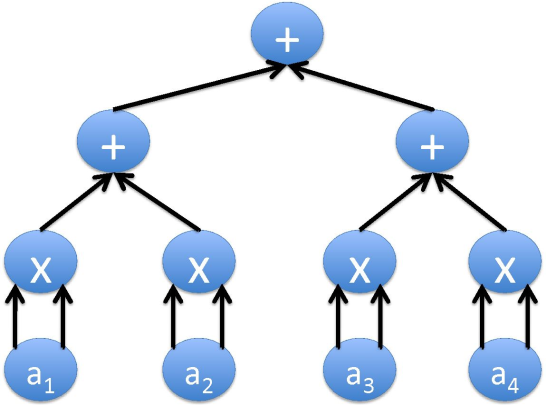
Recall: SNARKs for circuit-satisfiability

- Given: An arithmetic circuit C over $\mathbb{F}$ of size S and output y.
- P claims to know a w such that $C(x, w) = y$.
- For simplicity, let's take x to be the empty input.

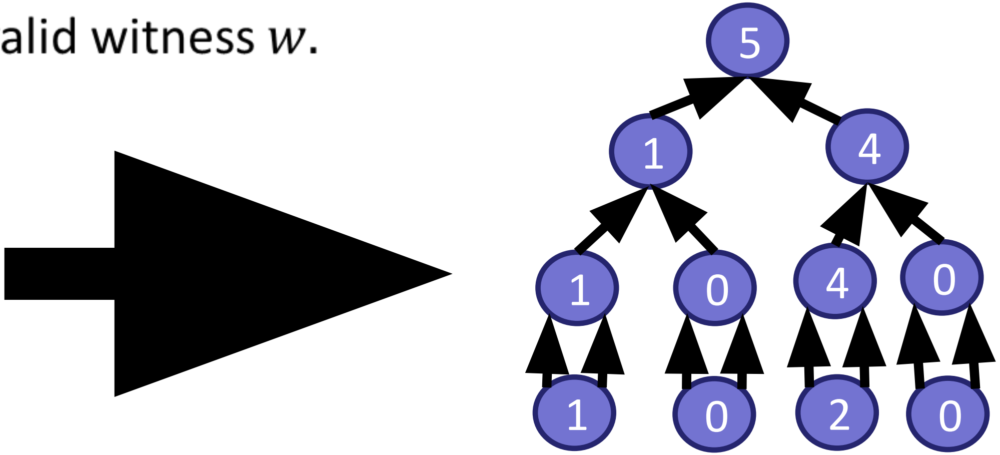


Recall: SNARKs for circuit-satisfiability

- A **transcript** T for C is an assignment of a value to every gate.
  - T is a **correct** transcript if it assigns the gate values obtained by evaluating C on a valid witness w.



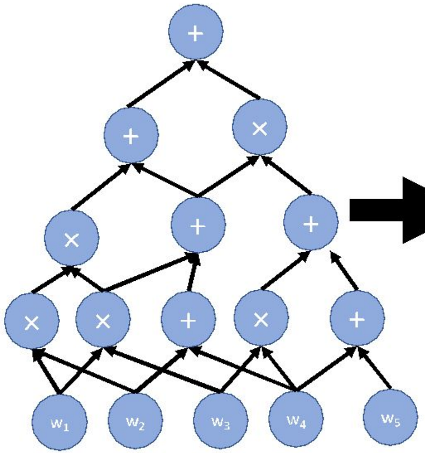
Circuit-SAT instance C



Correct transcript for C yielding output 5.

Viewing a transcript as a **function** with domain $\{0,1\}^{\log S}$

- Assign each gate in C a $(\log S)$-bit label and view T as a function mapping gate labels to $\mathbb{F}$.



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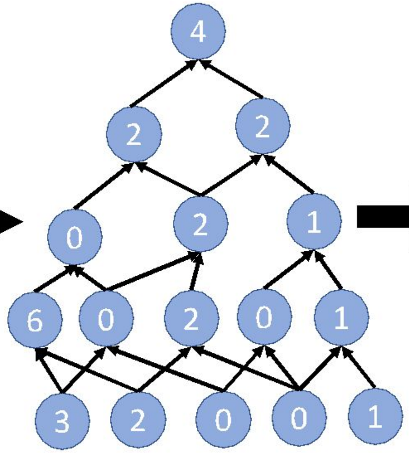

Correct transcript T for C

$T(0,0,0,0) = 3$
$T(0,0,0,1) = 2$
$T(0,0,1,0) = 0$
$T(0,0,1,1) = 0$
$T(0,1,0,0) = 1$
$T(0,1,0,1) = 6$
$T(0,1,1,0) = 0$
$T(0,1,1,1) = 2$
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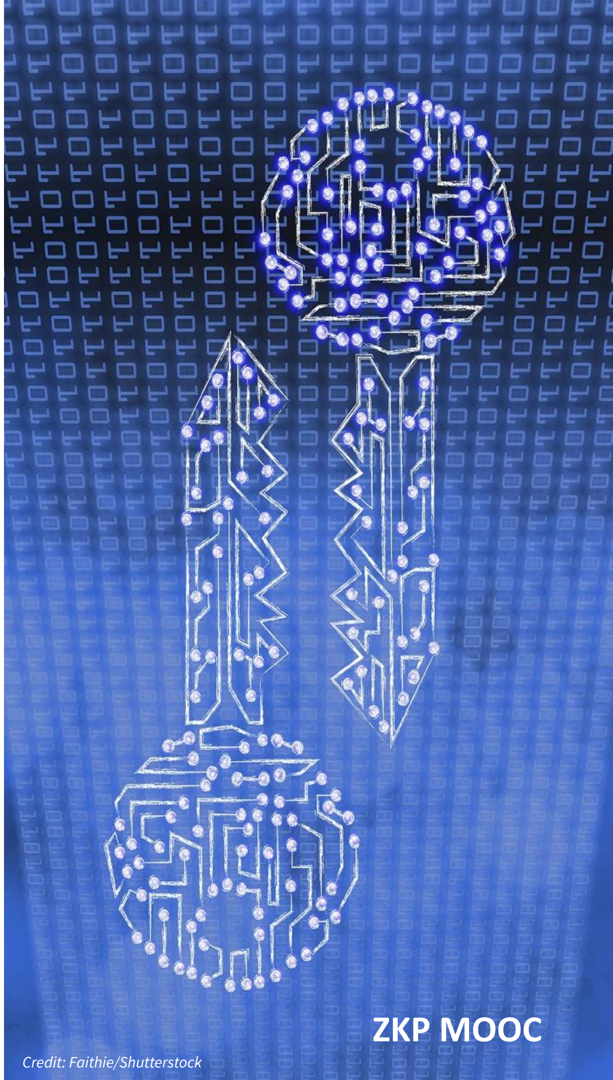
$T(1,1,0,1) = 2$
$T(1,1,1,0) = 2$
$T(1,1,1,1) = 4$

The polynomial IOP underlying the SNARK



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The start of the polynomial IOP

- Assign each gate in C a $(\log S)$-bit label and view T as a function mapping gate labels to $\mathbb{F}$.
- P's first message is a $(\log S)$-variate polynomial h claimed to **extend** a correct transcript T, which means:

$$h(x) = T(x)\ \forall\ x \in \{0,1\}^{\log S}.$$

V needs to check this, but is only able to learn a few evaluations of h.

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Intuition for why h is a useful object for P to send

- Think of h as a **distance-amplified encoding** of the transcript T.
- The domain of T is $\{0,1\}^{\log S}$. The domain of h is $\mathbb{F}^{\log S}$, which is vastly bigger.

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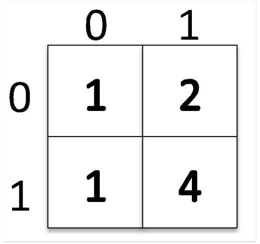
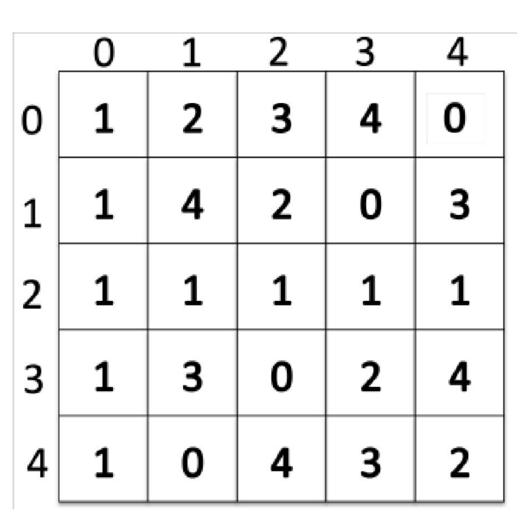


	0	1
0	1	2
1	1	4

All four evaluations of a function T mapping $\{0,1\}^2$ to $\boldsymbol{F}_5$



	0	1	2	3	4
0	1	2	3	4	0
1	1	4	2	0	3
2	1	1	1	1	1
3	1	3	0	2	4
4	1	0	4	3	2

All 25 evaluations of the multilinear polynomial h that extends T, one for each element of $\boldsymbol{F}_5 \times \boldsymbol{F}_5$

Intuition for why h is a useful object for P to send

- Think of h as a **distance-amplified encoding** of the transcript T.
- The domain of T is $\{0,1\}^{\log S}$. The domain of h is $\mathbb{F}^{\log S}$, which is vastly bigger.
- Schwartz-Zippel: If two transcripts T, T' disagree at even a **single** gate value, their extension polynomials h, h' disagree at **almost all** points in $\mathbb{F}^{\log S}$.
  - Specifically, a $1 - \log(S)\ /|\mathbb{F}|$ fraction.
- Distance-amplifying nature of the encoding will enable V to detect even a single "inconsistency" in the entire transcript.

Reminder: the start of the polynomial IOP

- P's first message is a $(\log S)$-variate polynomial h claimed to **extend** a correct transcript T, which means:

$$h(x) = T(x) \; \forall \, x \in \{0, 1\}^{\log S}.$$

- V needs to check this, but is only able to learn a few evaluations of h.

Two-step plan of attack

- 1. Given any $(\log S)$-variate polynomial h, identify a related $(3\log S)$-variate polynomial g_h such that:

  h **extends** a correct transcript $T \Leftrightarrow g_h(a, b, c) = 0 \ \forall (a, b, c) \in \{0,1\}^{3 \log S}$.

  - Moreover, to evaluate $g_h(r)$ at any input r, suffices to evaluate h at only 3 inputs.

  2. Design an interactive proof to check that $g_h(a, b, c) = 0 \ \forall (a, b, c) \in \{0,1\}^{3 \log S}$.

  - In which V only needs to evaluate $g_h(r)$ at one point r.

Step 1 of the plan

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  - And to evaluate $g_h(r)$ at any r, suffices to evaluate h at only 3 inputs.

Proof sketch (simplification): Define $g_h(a,b,c)$ via:

$$\widetilde{add}(a,b,c)\cdot\Big(h(a)-\big(h(b)+h(c)\big)\Big)+\widetilde{mult}(a,b,c)\cdot\big(h(a)-h(b)\cdot h(c)\big).$$

$g_h(a,b,c)=h(a)-\big(h(b)+h(c)\big)$ if a is the label of a gate that computes the sum of gates b and c.

$g_h(a,b,c)=h(a)-h(b)\cdot h(c)$ if a is the label of a gate that computes the product of gates b and c.

$g_h(a,b,c)=0$ otherwise.

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  3. $g_h(a,b,c) = 0$ otherwise.

Step 2: A Hint

- How to check that $g_h(a,b,c) = 0 \ \forall (a,b,c) \in \{0,1\}^{3\log S}$?
  - With V only evaluating g_h at a **single** point?
- Imagine for a moment that g_h were a **univariate** polynomial $g_h(X)$.
  - And rather than needing to check that g_h vanishes over input set $\{0,1\}^{3\log S}$, we needed to check that g_h vanishes over some set $H \subseteq \mathbb{F}$.
- Fact: $g_h(x) = 0$ for all $x \in H \Leftrightarrow$ g_h is divisible by $Z_H(x) \coloneqq \prod_{a \in H}(x - a)$.
  - Z_H is called the vanishing polynomial for H.
- Polynomial IOP:
  - P sends a polynomial q such that $g_h(X) = q(X) \cdot Z_H(X)$.
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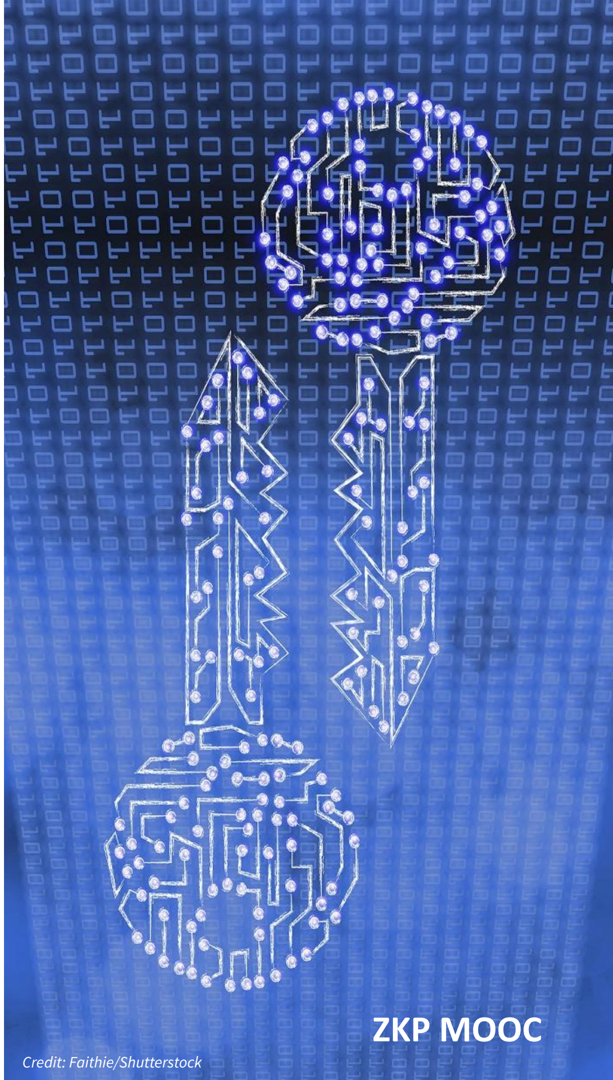
The actual protocol

- Previous slide doesn't actually work.
  - g_h is not univariate, it has $3 \log S$ variables.
- Also, having P find and send the quotient polynomial is expensive.
  - In the final SNARK, this would mean applying polynomial commitment to additional polynomials.
  - This is what Marlin, PlonK, and Groth16 do.
- Solution: use the sum-check protocol [LFKN90].
  - Handles multivariate polynomials.
  - Doesn't require P to send additional large polynomials.

Recall sum-check



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Sum-check protocol: a reminder

- Goal: compute the quantity:

$$\sum_{b_1 \in \{0,1\}} \sum_{b_2 \in \{0,1\}} \dots \sum_{b_\ell \in \{0,1\}} g(b_1, \dots, b_\ell).$$

- Proof length is roughly the total degree of g.
- Number of rounds is ℓ.
- V time is roughly the time to evaluate g at a single randomly chosen input.
- To run the protocol, V doesn't even need to "know" what polynomial g is being summed, so long as it knows $g(r)$ for a randomly chosen input $r \in \mathbb{F}^\ell$.

The polynomial IOP for circuit-satisfiability

- How to check that $g_h(a,b,c) = 0 \; \forall (a,b,c) \in \{0,1\}^{3 \log S}$?
  - With V only evaluating g_h at a **single** point?
- General idea (working over the integers instead of $\mathbb{F}$):
  - V checks this by running sum-check protocol with P to compute:

$$\sum_{a,b,c \in \{0,1\}^{\log s}} g_h(a,b,c)^2.$$

  - If all terms in the sum are 0, the sum is 0.
  - If working over the integers, any non-zero term in the sum will cause the sum to be strictly positive.

The polynomial IOP for circuit-satisfiability

- How to check that $g_h(a,b,c) = 0 \ \forall (a,b,c) \in \{0,1\}^{3\log S}$?
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- General idea (working over the integers instead of $\mathbb{F}$):
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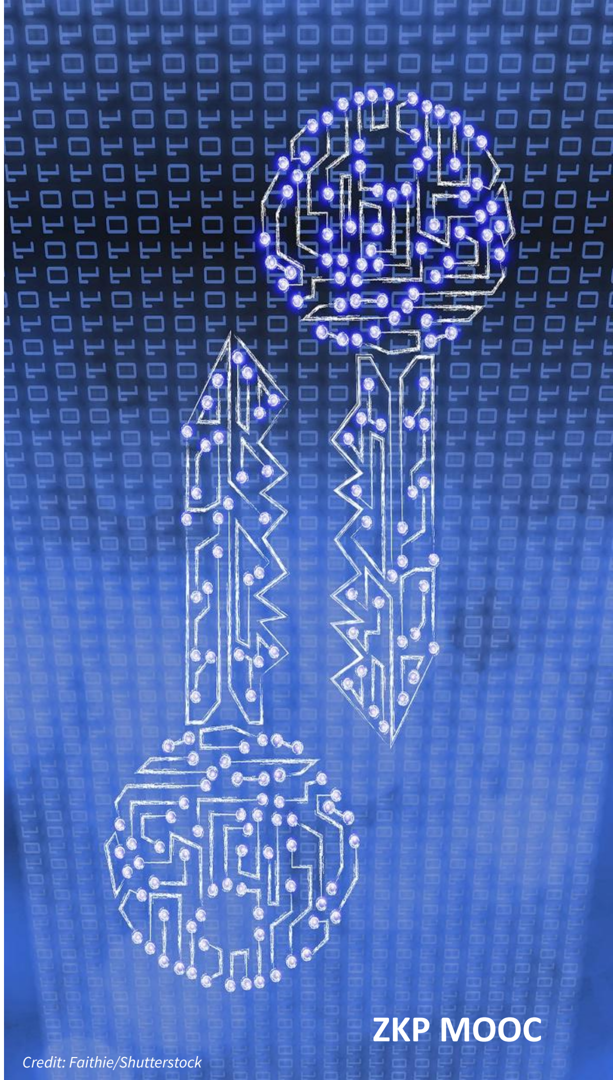
- At end of sum-check protocol, V needs to evaluate $g_h(r_1, r_2, r_3)$.
  - Suffices to evaluate $h(r_1), h(r_2), h(r_3)$.
  - Outside of these evaluations, V runs in time $O(\log S)$.
  - P performs $O(S)$ field operations given a witness w.

END OF LECTURE


ZKP MOOC

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